Advances In Hybrid Polynomial Basis Functions: Novel Numerical Approach for Partial Differential Equations

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Abstract

This scholarly investigation concentrates on the formulation of an optimized numerical methodology predicated upon the amalgamation of two polynomial basis functions, aimed at resolving a specific subset of linear and nonlinear Partial Differential Equations (PDEs). In this methodological framework, a solution derived from power series is utilized, incorporating Chebyshev and Hermite polynomials to satisfy the particular requirements imposed by the PDE. Subsequently, by substituting the series solution into the designated PDE and employing appropriate collocation points, a linear system of algebraic equations featuring indeterminate hybridization coefficients was established, with its resolution achieved through the Gauss elimination method (GEM) utilizing numerical computation software. Moreover, various discretization schemes were investigated to elucidate the manner in which the results fluctuate in response to variations in the allocation of collocation points throughout the domain. Two illustrative cases were scrutinized employing the numerical technique to assess the method's efficacy in terms of reliability, effectiveness, and precision. The findings obtained were subjected to benchmarking and validation against established results within the existing body of literature.

Keywords: Chebyshev Polynomials, Hermite Polynomials, Linear and Nonlinear Partial Differential Equations, Collocation Points, Gauss Elimination Method

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Introduction

A foundational mathematical construct recognized as partial differential equations (PDEs) serves to elucidate a diverse array of physical, engineering, and scientific phenomena. These equations are indispensable instruments for the analysis of complex systems that exhibit variation in both spatial and temporal domains, as they provide a robust methodology for modeling the alterations of quantities in relation to multiple independent variables.

The examination of PDEs boasts a rich and venerable historical lineage, tracing back to the pioneering contributions of eminent mathematicians such as Leonhard Euler and Joseph-Louis Lagrange. Scientists use partial differential equations (PDEs) as essential tools for various applications include fluid dynamics, thermal conduction and both quantum physics and image processing since their initial discovery in history. Their extensive application in both research and engineering domains is attributable to their ability to encapsulate the fundamental principles governing the evolution of ongoing physical phenomena.

A pivotal element in the resolution of PDEs involves articulating their solutions in terms of orthogonal polynomials. The defining characteristic of orthogonal polynomials as a mathematical function type involves their orthogonal relationship throughout specific intervals when utilizing distinct weighting functions. The significance of partial differential equations in the realm of scientific computing is paramount. Their historical importance encompasses the modeling of natural phenomena and the bridging of the divide between pure and applied mathematics, thereby establishing a solid foundational presence in various scientific disciplines. The main identifying feature of every mathematical orthogonal polynomial type exists in its orthogonal function relationship while utilizing different weighting functions in designated intervals.

Adenipekun et al (2024) combined Chebyshev and Legendre shifted polynomial basis functions to obtain approximate solutions for nonlinear Partial Differential Equations giving superior results against current studies. Muhammad Arfan, et al (2022) researched an approximate calculation technique for fractional differential equations (FDEs) based on Hermite polynomials under the Caputo fractional operator. A system of algebraic equations arises from the approach which enables Gauss elimination to solve the equations and produce the desired solution thereby demonstrating enhanced effectiveness of this method. Caporale and Carrato (2010) applied Chebyshev polynomials for the approximate solution of PDEs. Galal, I. E., et al (2024) introduced a boundary value problem solution method through collocations that works well with high-order features. Akyuz-Dascioglu (2009) used the idea of polynomial approximation in terms of Chebyshev for the high-order PDEs with complex conditions. The concept of employing the Hermite-based block approach for the numerical solution of second-order nonlinear elliptic partial differential equations utilizing the interpolation and collocation procedures was examined by Emmanuel, O. A., et al. in 2023.

By establishing a new Hilbert space that meets the beginning and boundary requirements, Atta and Youssri (2022) concentrated on an approximation spectral technique for the nonlinear time-fractional partial integro-differential equation with a weakly singular kernel.

Luo *et al.* (2017) used the Barycentric rational-collocation approach to address a set of nonlinear parabolic PDEs. Dhiman and Tamsir (2018) explored a collocation method that utilized a modified version of cubic B-spline trigonometric functions to address the Fisher's reaction diffusion equation.

Hongchun Wu, Yulan Wang and Wei Zhang (2018) developed a meshless barycentric interpolation collocation method (MBICM) for the solution of a class of partial differential equations (PDEs). The approximate solutions were in comparison with some existing literature and prove better. Ayatollah, Yari, and colleagues (2020) developed a numerical approach for solving a class of fractional optimum control problems that relies on the Hermite polynomial approximation.

Yuksel and Sezer (2013) introduced an approximating linear second-order PDEs approach with complex boundary conditions using Chebyshev scheme expansion.

Karunakar and Chakraverty (2019) employed a highly effective technique relying on the shifted Chebyshev polynomials to tackle PDEs. They selected a series of power solution involving the shifted Chebyshev polynomials to ensure compliance with the specified conditions. Kehali, S. and Khirani, A. (2024) made the use of Hermite polynomials to solve the Volterra-Fredholm integral equations of second kind numerically. Samaneh *et al.* (2020) developed a method to investigate the inverse problem in the estimation of time-dependent heat and temperature source in the context of the heat equation, which includes the Dirichlet boundary parameters and the integral over-determination condition. The solution approximation for this problem employs shifted Chebyshev polynomials as the foundation of Tau method depending on the operational matrices in form of Chebyshev.

Here, we want to look at the area of hybridizing the Chebyshev and Hermite polynomial basis functions.

Preliminaries

The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_k(v)$. The first kind of Chebyshev polynomials is denoted $T_k(v)$. The Chebyshev polynomial of the first kind $T_k(z)$ can be defined by the contour integral

$$T_k(z) = \frac{1}{4\pi i} \oint \frac{\left(1-t^2\right) t^{-k-1}}{\left(1-2tz+t^2\right)} \, dt \tag{1}$$

Where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

The polynomial of nth degree with unity as the leading coefficient in the range [-1, 1] was first introduced by the Russian mathematician Pafnuty Chebyshev.

$$T_k(v) = Cos\left(k\cos^{-1} v\right), \qquad k = 0,1,2,3,4\ldots \tag{2}$$

The recurrence formula can be used to find the Chebyshev polynomials;

$$T_{k+1}(v) = 2vT_k(v) - T_{k-1}(v), \qquad k = 1,2,3,4\ldots \tag{3}$$

The first few Chebyshev polynomials of the first kind are

$$T_0(v) = 1$$
$$T_1(v) = v$$
$$T_2(v) = 2v^2 - 1$$
$$T_3(v) = 4v^3 - 3v$$
$$T_4(v) = 8v^4 - 8v^2 + 1$$
$$T_5(v) = 16v^5 - 20v^3 + 5v$$

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On the other hand, Hermite polynomials are a set of orthogonal polynomials that play a significant role in various fields such as mathematics, physics, and engineering. These polynomials are named after Charles Hermite, a French mathematician. They're well-known for their use in probability, quantum mechanics, and numerical analysis. Essentially, these polynomials are solutions to Hermite's differential equation and come in two types: the physicist's Hermite polynomials and the probabilist's Hermite polynomials. Hermite polynomials are useful tools in math with a variety of uses in both theory and practice. They're especially important for solving differential equations and have applications in fields like quantum mechanics and statistical mechanics. These polynomials are written as $H_k(v)$ and are defined using a specific recurrence relation.

$$H_{k+1}(v) = 2vH_k(v) - 2kH_{k-1}(v) \tag{4}$$

For k ≥ 1, with initial conditions $H_0(v) = 1$ and $H_1(v) = 2v$. These polynomials meet this differential equation: $H_k^{//}(v) - 2vH_k^{/}(v) + 2kH_k(v) = 0$

The Hermite polynomial $H_k(z)$ can be defined by the contour integral;

$$H_k(z) = \frac{k!}{2\pi i}\oint e^{-t^2+2tz}\, t^{-k-1}\, dt \tag{5}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

The first few Hermite polynomials are

$$H_0(v) = 1$$
$$H_1(v) = 2v$$
$$H_2(v) = 4v^2 - 2$$
$$H_3(v) = 8v^3 - 12v$$
$$H_4(v) = 16v^4 - 48v^2 + 12$$
$$H_5(v) = 32v^5 - 160v^3 + 120v$$

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3.0 Methodology

3.1 The hybridization collocation method

Here, we describe the approach to solve partial differential equations (PDEs); the amalgamation of Chebyshev and Hermite polynomials. To exemplify the technique, we examine a partial differential equation in generals characterized by two distinct variables which are m and t; and w the dependent variable represented as follows:

$$H(w) = c(m,t) \tag{6}$$

Depending on the stipulated terms and circumstances:

$$w(m_0,t) = d(t) \quad and \quad w(m,t_0) = k(m) \tag{7}$$

where w is an unspecified function; c, d and k denote specified functions while m_0 and t_0 remain constants. To begin, we make the initial assumption of a two-variable power series solution expressed using the Chebyshev and Hermite basis polynomial functions, which must meet the criteria outlined in (8) as below

$$w(m,t) = \xi(m,t) + \eta(m,t) \sum_{q=0}^{N} \sum_{s=0}^{N} a_{qs} T_q^*(m) \tilde{\tilde{P}}_s(t) \tag{8}$$

It's significant to note $\xi(m,t)$ and $\eta(m,t)$ to be characteristics of both m and t, and their selection should such that (8) complies with the prescribed requirements set forth in condition (7). Then, we substitute the assumed solution (8) into (6), and from this process, we derive a residual equation denoted as $Q(m,t)=0$, which encompasses coefficients represented as a_{qs}. By employing appropriate points of collocation within $[0,1]$ *by* $[0,1]$; we formulate a system of the equation with the unspecified coefficients, denoted by a_{qs}

$$Hv = b \tag{9}$$

In this context, H stands as the well-established coefficient of the matrix, v represents column vector containing Chebyshev and Hermite coefficients, and b is the predetermined vector at the right-hand side (RHS). The very important thing to note here is that both H and b consist of real values determined through the utilization of the points of collocation within the equation of residuals $L(m,t)=0$.

Now, $M_p{}^2$ as collocation points become a requisite when we consider M_p polynomial where $M_p = N+1$. Subsequently, the set of equations for determining the unspecified coefficients are established, followed by their substitution into (8), we get the approximate solution of (6).
The discretization of the domain represents a significant consideration and should be noted. As a result, different discretization patterns will be considered before the best one is finally selected to produce results that are acceptable.

Numerical Examples and their Results

Example 1: Consider a linear partial differential equation of the form;

$$\frac{\partial^2 H}{\partial y^2} - 9\frac{\partial^2 H}{\partial x^2} - e^{x} + e^{-x} = 0 \tag{4.0.1}$$

Subject to the conditions:

$$H(x,0) = x \quad and \quad H_y(x,0) = \sin x \tag{4.0.1.1}$$

The theoretical solution of the given equation is given to be

$$H(x,y) = x + \frac{1}{3}\sin x \sin 3y - \frac{2}{9}\sinh x + \frac{2}{9}\sinh x \cosh 3y \tag{4.0.1.2}$$

Assume that the bivariate series solution for the linear PDE (4.0.1) as;

$$H(x,\, y) = x + y\sin(x) + \left(x\, y^2\right)\sum_{q=0}^{4}\sum_{s=0}^{4} a_{qs}\, T_q^*(x) P_s^*(y) \tag{4.0.2}$$

where $\omega(x,y) = x + y\sin x \quad and \quad \lambda(x,y) = x\,y^2$. Initial conditions (4.0.1.1 are satisfied by the anticipated solution 4.0.2). Now, differentiate (4.0.2) partially with relation to x and y to acquire the derivatives as;

$$\begin{aligned}
\frac{\partial^2 H}{\partial x^2} &= -y\sin x + 12xy^2\left(y^2-1\right)a_{22} + 12xy^2\left(y^3-3y\right)a_{23} + 12xy^2\left(y^4-6y^2+3\right)a_{24} + \\
&2y^2\left(12x^2-3\right)\left(y^2-1\right)a_{32} + 24x^2y^2\left(y^2-1\right)a_{32} + 2y^2\left(12x^2-3\right)\left(y^3-3y\right)a_{33} + 24x^2y^2 \\
&\left(y^3-3y\right)a_{33} + 2y^2\left(12x^2-3\right)\left(y^4-6y^2+3\right)a_{34} + 24x^2y^2\left(y^4-6y^2+3\right)a_{34} + 2y^2\left(y^2-1\right) \\
&\left(32x^3-16x\right)a_{42} + 2y^2\left(32x^3-16x\right)\left(y^3-3y\right)a_{43} + 2y^2\left(32x^3-16x\right)\left(y^4-6y^2+3\right)a_{44} + \\
&xy^2\left(96x^2-16\right)a_{40} + 2y^2\left(y^2-1\right)a_{12} + 2y^2\left(y^3-3y\right)a_{13} + 2y^2\left(y^4-6y^2+3\right)a_{14} \\
&+ xy^3(96x^2-16)a_{41} + 24x^2y^2a_{30} + 12xy^3a_{21} + 2y^3(12x^2-3)a_{31} + 2y^3(32x^3-16x)a_{41} \\
&+ 2y^2(12x^3-3)a_{30} + 2y^2(32x^3-16x)a_{40} + 12xy^2a_{20} + 24x^2y^3a_{31} + 2y^3a_{11} + 2y^2a_{10} + \\
&xy^2(96x^2-16)(y^2-1)a_{42} + xy^2(96x^2-16)(y^3-3y)a_{43} + xy^2(96x^2-16)(y^4-6y^2+3)a_{44}
\end{aligned} \tag{4.02.1}$$

and

$$
\begin{aligned}
\frac{\partial^2 H}{\partial y^2} &= xy^2\left(12y^2-12\right)a_{04}+2x\left(2x^2-1\right)a_{20}+2x\left(4x^3-3x\right)a_{30}+2x\left(8x^4-8x^2+1\right)a_{40} \\
&+6xy\left(2x^2-1\right)a_{21}+x^2y^2\left(12y^2-12\right)a_{14}+6xy\left(4x^3-3x\right)a_{31}+6xy\left(8x^4-8x^2+1\right)a_{41} \\
&+2x\left(2x^2-1\right)a_{02}+2x\left(y^3-3y\right)a_{03}+2x\left(y^4-6y^2+3\right)a_{04}+6xy^3\left(2x^2-1\right)a_{23}+6xy^3 \\
&\left(4x^3-3x\right)a_{33}+6xy^3\left(8x^4-8x^2+1\right)a_{43}+2x^2\left(y^2-1\right)a_{12}+2x^2\left(y^3-3y\right)a_{13}+2x^2 \\
&\left(y^4-6y^2+3\right)a_{14}+10xy^2\left(2x^2-1\right)a_{22}+10xy^2\left(4x^3-3x\right)a_{32}+10xy^2\left(8x^4-8x^2+1\right)a_{42} \\
&+2xa_{00}+10x^2y^2a_{12}+6xy^3a_{03}+6x^2ya_{11}+10xy^2a_{02}+6xya_{01}+4xy\left(3y^2-3\right)a_{03} \\
&+4xy\left(4y^3-12y\right)a_{04}+4x^2y\left(3y^2-3\right)a_{13}+4x^2y\left(4y^3-12y\right)a_{14}+2x^2a_{10}+xy^2 \\
&\left(2x^2-1\right)\left(12y^2-12\right)a_{24}+xy^2\left(4x^3-3x\right)\left(12y^2-12\right)a_{34}+xy^2\left(8x^4-8x^2+1\right)\left(12y^2-12\right)a_{44} \\
&+2x\left(2x^2-1\right)\left(y^2-1\right)a_{22}+2x\left(2x^2-1\right)\left(y^3-3y\right)a_{23}+2x\left(2x^2-1\right)\left(y^4-6y^2+3\right)a_{24} \\
&+2x\left(4x^3-3x\right)\left(y^2-1\right)a_{32}+2x\left(4x^3-3x\right)\left(y^3-3y\right)a_{33}+2x\left(4x^3-3x\right)\left(y^4-6y^2+3\right)a_{34} \\
&+2x\left(8x^4-8x^2+1\right)\left(y^2-1\right)a_{42}+2x\left(8x^4-8x^2+1\right)\left(y^3-3y\right)a_{43}+2x\left(8x^4-8x^2+1\right) \\
&\left(y^4-6y^2+3\right)a_{44}+4xy\left(2x^2-1\right)\left(3y^2-3\right)a_{23}+4xy\left(2x^2-1\right)\left(4y^3-12y\right)a_{24}+4xy \\
&\left(4x^3-3x\right)\left(3y^2-3\right)a_{33}+4xy\left(4x^3-3x\right)\left(4y^3-12y\right)a_{34}+4xy\left(8x^4-8x^2+1\right)\left(3y^2-3\right)a_{43} \\
&+4xy\left(8x^4-8x^2+1\right)\left(4y^3-12y\right)a_{44}+6x^2y^3 \qquad (4.02.2)
\end{aligned}
$$

Now, substitute (4.0.2.1) and (4.0.2.2) into (4.0.2) to have the residual form of equation $H(x,y)=0$. Then, collocation at twenty-five points since our N = 4, which in turns gives a system of twenty-five linear algebraic equations. Having solved the system by method of Gauss elimination or computational software, we have the unknowns as;

$a_{00} = 0.8145479389$, $a_{01} = 1.709176165$, $a_{02} = -1.224877796$, $a_{03} = 1.006146222$,
$a_{04} = -0.3209127867$, $a_{10} = 0.3292478285$, $a_{11} = -0.4757335546$, $a_{12} = 0.4785579422$,

$a_{13} = -0.1313157764$, $a_{14} = 0.05535719150$, $a_{20} = 0.03393631642$, $a_{21} = 0.09770000226$
$a_{22} = -0.05542446971$, $a_{23} = -0.02347730343$, $a_{24} = -0.00411584782$3, $a_{30} = 0.08536341407$
$a_{31} = -0.1211129138$, $a_{32} = 0.1248642276$, $a_{33} = -0.03332552358$, $a_{34} = 0.01463010342$
$a_{40} = -0.01636189362$, $a_{41} = 0.02174155860$, $a_{42} = -0.02656119063$, $a_{43} = 0.006698253473$
$a_{44} = -0.00340661598$8

Substituting all a'_{qs} in (4.0.2), we obtain an approximate form of solution for the differential

equation as thus;

$$
\begin{aligned}
H(x, y) &= -0.02028837769x^2y^2(y^2-1) - 0.3884567956xy^2(y^4-6y^2+3) + \\
&0.06432227671xy^2(2x^2-1) - 0.8284703469xy^2(y^2-1) - 0.005521053931xy^2(4x^3-3x) \\
&+ x + y\sin x + 0.007262710386xy^2(8x^4-8x^2+1) + 0.004987983988x^2y^2(y^3-3y) + \\
&0.6519750584xy^2(y^3-3y) - 0.01175334246xy^3(8x^4-8x^2+1) - 0.001662662874x^2y^2 \\
&(y^4-6y^2+3) + 0.1154023357xy^3(2x^2-1) + 0.007569543413xy^3(4x^3-3x) + \\
&0.5701672721xy^3 - 0.02002238184x^2y^2 + 0.02727544084x^2y^3 + 1.426176719xy^2 - \\
&0.1024171150xy^2(2x^2-1)(y^2-1) + 1.44879170710xy^2(2x^2-1)(y^3-3y) - \\
&0.02538588400xy^2(2x^2-1)(y^4-6y^2+3) + 0.001416710120xy^2(4x^3-3x)(y^3-3y) \\
&- 0.0004722368333xy^2(4x^3-3x)(y^4-6y^2+3) + 0.009022724752xy^2(8x^4-8x^2+1) \\
&(y^2-1) - 0.003152847462xy^2(8x^4-8x^2+1)(y^3-3y) + 0.001050949508xy^2 \\
&(8x^4-8x^2+1)(y^4-6y^2+3) - 0.005673720773xy^2(4x^3-3x)(y^2-1)
\end{aligned}
$$

But the exact solution of (10) is

$$H(x,\ y) = x + \frac{1}{3}\sin x \sin 3y - \frac{2}{9}\sin hx + \frac{2}{9}\sin hx\ \cosh 3y$$.

For table one, the enumerated error norms at different time is shown.

Table 1 shows the computations of the results when x = 0.5 at different points of discretization

x	y	Exact solution	Approximate solution at N = 2	Approximate solution at N = 3	Approximate solution at N = 4
0.5	0.001	0.5004799459	0.5004799458	0.5004799459	0.5004799458
0.5	0.003	0.5014429470	0.5014429458	0.5014429467	0.5014429472
0.5	0.005	0.5024100654	0.5024100595	0.5024100636	0.5024100657
0.5	0.007	0.5033812667	0.5033812508	0.5033812617	0.5033812674
0.5	0.009	0.5043565168	0.5043564841	0.5043565062	0.5043565187
0.5	0.01	0.5048456497	0.5048456055	0.5048456352	0.5048456527

Table 2 shows the error norms from table 1 results, also with the existing literature

x	y	Karunakar.et al (2019) at N = 3	Prospose d method at N = 2	Prospose d method at N = 3	Prospose d method at N = 4
0.5	0.001	6.16966E-04	1.0007E-10	1.00004E-10	1.00121E-10
0.5	0.003	1.85089E-03	1.19425E-09	3.01245E-10	2.00413E-10
0.5	0.005	3.08481E-03	5.91203E-09	1.78814E-09	3.00091E-10
0.5	0.007	4.31871E-03	1.59042E-08	5.01142E-09	7.00311E-10
0.5	0.009	5.55258E-03	3.27144E-08	1.05994E-08	1.89914E-09
0.5	0.01	6.16950E-03	4.41811E-08	1.45110E-08	3.00044E-09

Table 2 shows the error norms from table 1 results, also with the existing literature

4.1 Discussion of Results

It is evident from the table one above that the approximate results gotten with the current scheme perform well and it is in a satisfactory approval with the theoretical solution of Example 1. Collation points are picked within the boundary and have proven good based on the results obtained. This method also requires choosing different discretization patterns to obtain a better approximate solution for the PDEs. We could see from the solution that

Example 2: Examine the nonlinear partial differential equation featuring $\rho = -1$ and

$$z(v) = \left(\frac{\psi^2}{4}\right)v + v^2,$$

$$\frac{\delta^2 v}{\delta j^2} - \frac{\delta^2 v}{\delta c^2} - c^2 \sin^2\left(\frac{\psi}{2} j\right) = -z(v) \tag{10}$$

where $c \in (-1, 1), \quad j > 0$. (12) meets the conditions $v(c, 0) = 0, \dfrac{\delta v}{\delta j}(c, 0) = \dfrac{\psi}{2}c \quad where \quad c \in [-1, 1]$; as well as the Dirichlet boundary conditions

$$v(-1, j) = -\sin\left(\frac{\psi}{2} j\right), \; v(1, j) = \sin\left(\frac{\psi}{2} j\right) \; such\ that \;\; j \geq 0 \tag{11}$$

The theoretical solution for the differential equation is given to be $v(c, j) = c\sin\left(\dfrac{\psi}{2} j\right)$.

Assume that the bivariate series solution for the PDE (10) as;

$$v(c, j) = \frac{\psi}{2}c\,j + c^3 \sin\left(\frac{\psi}{2} j\right) - \frac{\psi}{2}c^3 j + \left(\frac{\psi}{2}c\,j^2 - \frac{\psi}{2} j^2 c^3\right)\sum_{q=0}^{4}\sum_{s=0}^{4} a_{qs} T_q^*(c) P_s^*(j) \tag{12}$$

where

$$\mu(c, j) = \frac{\psi}{2}c\,j + c^3 \sin\left(\frac{\psi}{2} j\right) - \frac{\psi}{2}c^3 j \quad and \quad \lambda(c, j) = \frac{\psi}{2}c\,j^2 - \frac{\psi}{2} j^2 c^3$$

Initial conditions (11 are satisfied by the anticipated solution 12). Now, differentiate (12) partially with relation to m and t to acquire the derivatives as . Following the same procedure as example 1, the unknown coefficients obtained ar

$$a_{00} = 478.5035528,\ a_{01} = -990.6016191,\ a_{02} = 656.5779175,\ a_{03} = -295.4105208,$$
$$a_{04} = 68.97997449,\ a_{10} = -820.3329039,\ a_{11} = 1689.889653,\ a_{12} = -1127.909615,$$
$$a_{13} = 503.9388152,\ a_{14} = -118.9451146,\ a_{20} = 512.7349080,\ a_{21} = -1057.867253$$
$$a_{22} = 703.8409328,\ a_{23} = -315.2593926,\ a_{24} = 74.07224721,\ a_{30} = -211.9573819$$
$$a_{31} = 434.5471487,\ a_{32} = -291.0379041,\ a_{33} = 129.2073025,\ a_{34} = -30.68110718$$
$$a_{40} = 50.50973461,\ a_{41} = -104.9696197,\ a_{42} = 68.83106770,\ a_{43} = -31.18168814$$
$$a_{44} = 7.187344490$$

Substituting all $a_{qs}^{/}$ in (12), we obtain an approximate form of solution for the differential

Substituting all $a_{qs}^{/}$ in (12), we obtain an approximate form of solution for the differential
Equations as thus;

$$\begin{aligned}
v(c,j) &= -239.2517764\psi c^3 j^2 + \frac{1}{2}\psi c j - \frac{1}{2}\psi c^3 j + 239.2517764\psi c j^2 + 1689.889653 \\
&\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right) c j - 1127.909615 c\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(j^2 - 1\right) - 118.9451146 c \\
&\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(j^4 - 6 j^2 + 3\right) + 503.9388152 c\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(j^3 - 3 j\right) \\
&- 1057.867253 j\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(2c^2 - 1\right) - 820.3329039 c\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right) \\
&+ 68.97997449\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(j^4 - 6 j^2 + 3\right) - 990.6016191 j\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right) \\
&+ 656.5779175\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(j^2 - 1\right) - 295.4105208\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(j^3 - 3 j\right) \\
&+ 512.7349080\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(2c^2 - 1\right) + 68.83106770\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right) \\
&\left(8c^4 - 8c^2 + 1\right)\left(j^2 - 1\right) + 703.8409328\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(2c^2 - 1\right)\left(j^2 - 1\right) - 315.2593926 \\
&\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(2c^2 - 1\right)\left(j^3 - 3 j\right) + 74.07224721\left(\frac{1}{2}\psi c j^2 - \frac{1}{2}\psi c^3 j^2\right)\left(2c^2 - 1\right)
\end{aligned}$$

$$\left(j^4-6j^2+3\right)+434.5471487\,j\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)\left(4c^3-3c\right)-291.0379041$$
$$\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)\left(4c^3-3c\right)\left(j^2-1\right)+129.2073025\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)\left(4c^3-3c\right)$$
$$\left(j^3-3j\right)-30.68110718\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)\left(4c^3-3c\right)\left(j^4-6j^2+3\right)+c^3\sin\left(\frac{1}{2}\psi j\right)$$
$$-104.9696197\,j\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)\left(8c^4-8c^2+1\right)-211.9573819\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)$$
$$\left(4c^3-3c\right)+50.50973461\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)\left(8c^4-8c^2+1\right)-31.18168814$$
$$\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)\left(8c^4-8c^2+1\right)\left(j^3-3j\right)+7.187344490\left(\frac{1}{2}\psi c\,j^2-\frac{1}{2}\psi c^3 j^2\right)$$
$$\left(8c^4-8c^2+1\right)\left(j^4-6j^2+3\right)$$

But the theoretical solution of (15) is $v(c,\ j)=c\sin\left(\frac{\psi}{2}j\right)$. Table two below enumerates the errors with various points, j.

Table 2: Comparison of errors norms when c = 1 at various points (j)

c	j	Hongchun *et. al.*(2018)	Proposed method at N = 2	Proposed method at N = 3	Proposed method at N = 4
1.0	0.1	1.5457E-12	2.01234E-11	1.54879E-18	9.11245E-23
1.0	0.3	2.3010E-13	7.02543E-12	9.14842E-19	5.44123E-24
1.0	0.5	7.6876E-13	6.00124E-12	3.66411E-19	8.88058E-24
1.0	0.7	2.1556E-12	4.02541E-12	8.88126E-18	1.55461E-23
1.0	1.0	5.8307E-10	8.10453E-11	5.00124E-18	9.99548E-23

From the table above, it is clearly observed that the approximate results obtained using the present scheme perform favourably well and it is in a satisfactory approval with the theoretical solution of the example two. Collation points are picked within the boundary and have proven good based on the results obtained. This method also requires choosing different discretization patterns to obtain the better approximate results for the PDEs. We could see from the solution that pattern at different schemes (N = 2, N = 3 and N = 4) were observed, the scheme performed well but at N = 3 and N = 4 the results performed better here than the existing one.

5.0 Conclusion and Remarks

A numerical approach combines with the method of collocation, based on hybridizing the Chebyshev and Hermite basis polynomial functions was successfully applied to solve a set of linear and nonlinear PDEs. With this approach, a set of solutions in terms of the hybridized polynomials was assumed and satisfied the required conditions of the PDE. Comparing the approach with the previous methods described in the literature, the adjustment to the assumption has been successful in yielding an approximative solution with fewer terms. The numerical computation adopts the use of collocation methods while the convergence was established using a tolerance closer to the exact. Two examples were considered, and their results were in comparison with the available literatures. The outcomes demonstrated the effectiveness of the suggested approach and its ability to achieve convergence with fewer terms. However, it is recommended that more than two dimensional nonlinear PDEs with complex boundary conditions or irregular domains should be considered.

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